

EXACT SOLUTIONS IN WEYL-CUBED GRAVITY

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Abstract

A unitary gravitational action up to third order of curvature in which respects to the holographic a -theorem has been constructed in [5]. In particular, its third order term is just the Weyl-cubed term in four dimensions.

In this paper, we study this theory and find some its exact classical solutions. We show that the theory admits conformally flat, Lifshitz, Schrödinger and also hyperscaling-violating backgrounds as the solutions of equations of motion. Our analysis has been done for the pure Weyl-cubed gravity, Einstein plus Weyl-cubed term and gravity with matter.

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1 Introduction

Extended theories of gravity have attracted serious attempts during the last decades. In fact, these theories have opened new windows for studying some unsolved problems in Einstein gravity such as quantization of metric field. Moreover, these extended theories are good laboratories for testing some other aspects of theoretical physics for example AdS/CFT [1], and shed also lights on other areas of physics i.e. condensed matter physics.

Several approaches have been used to construct extended gravity see, for example, [2, 3]. One of the main difference between these approaches is the method one applies to obtain the action of the theory. For example, one may add new terms in action up to Curvature-squared order. It was shown that this extension of Einstein gravity is perturbatively renormalizable [4] but the theory contains massive spin 2 ghostlike mode and massless graviton mode.

Among the several interesting prescriptions for finding the action of extended gravity, the author of [5] were able to calculate an action up to cubic order of curvature using a concept which is called "the holographic a-theorem". The resulting action is given by².

$$I_G = \frac{1}{2l_p^{d-1}} \int d^{d+1}x \sqrt{-g} \left(\frac{d(d-1)}{L^2} \alpha + R + \mu L^2 \mathcal{X} + \beta L^4 \mathcal{Z} \right), \quad (1)$$

where \mathcal{X} is the four-dimensional Euler density [2],

$$\mathcal{X}_4 = R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2. \quad (2)$$

\mathcal{Z} contains curvature-cubed interactions and it was argued that a three-parameter family of unitary R^3 interactions would exist. The first is the cubic Lovelock interaction which is proportional to the six-dimensional Euler density \mathcal{X}_6

$$\mathcal{X}_6 = \frac{1}{8} \varepsilon_{abcdef} \varepsilon^{ghijkl} R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \quad (3)$$

²Some other approaches have been introduced for calculating the third order gravity see, for example, [8]

The second is the quasitopological interaction \mathcal{Z}_{d+1} ,

$$\begin{aligned}\mathcal{Z}_{d+1} = & R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + \frac{1}{(2d-1)(d-3)} \left(\frac{3(3d-5)}{8} R_{abcd} R^{abcd} R \right. \\ & - 3(d-1) R_{abcd} R^{abc}{}_e R^{de} + 3(d-1) R_{abcd} R^{ac} R^{bd} \\ & \left. + 6(d-1) R_a{}^b R_b{}^c R_c{}^a - \frac{3(3d-1)}{2} R_a{}^b R_b{}^a R + \frac{3(d-1)}{8} R^3 \right). \quad (4)\end{aligned}$$

Two other basis are constructed from the Weyl tensor

$$\mathcal{W}_1 = W_a{}^c{}_b{}^d W_c{}^e{}_d{}^f W_e{}^a{}_f{}^b, \quad \mathcal{W}_2 = W_{ab}{}^{cd} W_{cd}{}^{ef} W_{ef}{}^{ab}. \quad (5)$$

These four basis are not all independent for $d \geq 6$ due to the following relation

$$\mathcal{Z}_{d+1} = \mathcal{W}_1 + \frac{3d^2 - 9d + 4}{8(2d-1)(d-3)(d-4)} (\mathcal{X}_6 + 8\mathcal{W}_1 - 4\mathcal{W}_2). \quad (6)$$

Thus, we have a three-parameter family of *unitary* cubic interactions. For $d = 5$, \mathcal{Z}_6 is not defined and so \mathcal{X}_6 , \mathcal{W}_1 and \mathcal{W}_2 are the bases. For $d < 5$, $\mathcal{X}_6 = 0$ and by using Schouten identity, one has $\mathcal{W}_1 = \mathcal{W}_2$ [6]. Thus, in four dimension, we have a one-parameter family of interactions $\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2$.

Investigation of classical solutions of this cubic theory of gravity is of interest from several point of view. Indeed, first, one may ask weather the well-known solutions of the Einstein gravity are also the solutions of the cubic gravity or not. Moreover, one may search for backgrounds in which are absent in pure Einstein gravity [5, 7, 8, 9, 10]. Two classes of such solutions are Lifshitz and Schrödinger geometries. The Lifshitz background is given by [11]

$$ds^2 = -\frac{r^{2z}}{L^2} dt^2 + \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} d\vec{x}^2, \quad (7)$$

where z is the dynamical exponent and exhibits space and time scale differently

$$t \mapsto \lambda^z t, \quad r \mapsto \lambda^{-1} r, \quad \vec{x} \mapsto \lambda \vec{x}. \quad (8)$$

Also, the line element of Schrödinger geometry is [12]

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{2dt dx + dy^2}{r^2} \right), \quad (9)$$

Its scaling property is as follows

$$t \mapsto \lambda^z t, \quad r \mapsto \lambda r, \quad x \mapsto \lambda^{2-z} x, \quad y \mapsto \lambda x, \quad z \neq 1. \quad (10)$$

Beside of having various interesting geometrical and physical properties, these non-relativistic solutions have important applications in condensed matter physics [12, 13, 14].

A generalization of these geometries are backgrounds which are conformally related to these metrics [13]

$$d\tilde{s}^2 = r^{-2\frac{\theta}{d-1}} ds^2. \quad (11)$$

Here θ is called the hyperscaling violation exponent. Recalling AdS/CFT, the non-zero θ means that in dual field theory the hyperscaling violates and the entropy scales as $T^{\frac{d-\theta-1}{z}}$.

In this paper our aim is to find such classical solutions for the cubic theory in four dimensions. Recalling the previous discussions, it is enough to consider $\mathcal{W} = \mathcal{W}_1$ term at cubic order in four dimensions. Note also that in four dimensions the \mathcal{X}_4 is purely topological and does not contribute in the equations of motion.

The paper is organized as follows. In the next section, we will briefly introduce the cubic gravity. In section 2, we derive the equations of motion for the Weyl-cubed gravity. After that, in section 3, we study the possible classical solutions of the theory. In particular, we will study the pure Weyl-cubed gravity, Einstein plus Gauss-Bonnet plus Weyl-cubed gravity and gravity with matter separately and will find some known solutions such as conformally flat, Lifshitz and Schrödinger metrics.

Let us set $L = 1$ and define $\alpha = -\frac{\theta}{d-1}$ in the rest of the paper.

2 Equation of Motion and Solution

Let us consider the action as

$$I = I_G + I_M \quad (12)$$

where I_G denotes the gravitational part of the action in which its cubic term \mathcal{Z} contains only³ $\mathcal{W} = \mathcal{W}_1$ and I_M is the matter part of the action

$$I = -\frac{1}{2l_p^{d+1}} \int d^{d+1}x \sqrt{-g} (d(d-1)\Lambda + R + \mu\mathcal{X}_4 + \beta\mathcal{W}) + I_M. \quad (13)$$

Varying the action with respect to metric gives us the equations of motion as follows

$$G_{\mu\nu} - \frac{1}{2}d(d+1)\Lambda g_{\mu\nu} + \mu E_{\mu\nu} + \beta\mathcal{G}_{\mu\nu} = 2\frac{l_p^{d-1}}{\sqrt{-g}} \frac{\delta I_M}{\delta g_{\mu\nu}}, \quad (14)$$

where $G_{\mu\nu}$ is the Einstein tensor and $E_{\mu\nu}$ and $\mathcal{G}_{\mu\nu}$ are given by

$$E_{\mu\nu} = -\frac{1}{2}\mathcal{X}_4 g_{\mu\nu} + 2R_\mu^{\theta\sigma\gamma} R_{\nu\theta\sigma\gamma} - 4R_{\mu\theta\nu\gamma} R^{\theta\gamma} - 4R_{\mu\theta} R_\nu^\theta + 2RR_{\mu\nu}, \quad (15)$$

$$\begin{aligned} \mathcal{G}_{\mu\nu} = & -\frac{1}{2}W_{\rho\theta\sigma\gamma} F^{\rho\theta\sigma\gamma} g_{\mu\nu} + 6W_{\mu\theta\sigma\gamma} F_\nu^{\theta\sigma\gamma} \\ & - 3R_{\nu\theta\sigma\gamma} F_\mu^{\theta\sigma\gamma} - \frac{12}{d(d-1)}(dR_{\theta\sigma} - Rg_{\theta\sigma})F_\mu^{\theta\sigma}{}_\nu + \frac{6}{d(d-1)}R_{\mu\nu}F_{\rho\theta}^{\rho\theta} \\ & - 6\nabla_\theta\nabla_\sigma F_\mu^{\theta\sigma}{}_\nu - \frac{6}{d-1}\nabla_\rho\nabla_\gamma F^{\rho\theta}{}_\theta{}^\gamma g_{\mu\nu} - \frac{12}{d-1}\nabla_\rho\nabla_{(\mu}F^{\rho\theta}{}_{\theta\nu)} \\ & - \frac{6}{d(d-1)}\nabla_\mu\nabla_\nu F_{\rho\theta}^{\rho\theta} - \frac{6}{d-1}\nabla^2 F_{\mu\theta}^{\theta}{}_\nu + \frac{6}{d(d-1)}\nabla^2 F_{\rho\theta}^{\rho\theta} g_{\mu\nu} \end{aligned} \quad (16)$$

and it was used the following definition in (16)

$$F_{\rho\theta\sigma\gamma} = W_{\rho\theta}{}^{\epsilon\kappa} W_{\epsilon\kappa\sigma\gamma}. \quad (17)$$

It is notable that the above expression for $\mathcal{G}_{\mu\nu}$ can be simplified in four dimensions using the following relations [15]

$$W_{\rho\theta\sigma\gamma} W^{\rho\theta\sigma\gamma} = R_{\rho\theta\sigma\gamma} R^{\rho\theta\sigma\gamma} - 2W_{\rho\theta} W^{\rho\theta} + \frac{1}{3}R^2 \quad (18)$$

$$F_{\mu\theta}^{\theta}{}_\nu = -W_{\mu\theta\sigma\gamma} W_\nu^{\theta\sigma\gamma} = -\frac{1}{4}W_{\rho\theta\sigma\gamma} W^{\rho\theta\sigma\gamma} g_{\mu\nu}. \quad (19)$$

³In this section, we present the action and equations of motion for general $d+1$ dimensions but we will obtain the solutions in 4 dimension. Thus we neglect the \mathcal{Z}_{d+1} term in (13).

3 Solution

In this section, we would like to find the solutions of the equations of motion. We will first consider the pure Weyl-cubed gravity and try to obtain the vacuum solutions i.e. the solutions of $\mathcal{G}_{\mu\nu} = 0$. After that, we add the usual Einstein-Hilbert term to the action. At the end, we will consider the full action (12) with a particular form of matter action and discuss about the possible solutions of the theory.

3.1 Pure Weyl-Cubed Gravity

First of all, we would like to investigate the pure Weyl-cubed gravity. It is notable that the coefficient of \mathcal{W} term in the action (13) has dimension $(mass)^{d-5}$ which means that this term is *super-renormalizable* in $d = 3, 4$ dimensions, *renormalizable* in $d = 5$ dimension and is *non-renormalizable* for $d > 5$. In particular, in four dimension the pure Weyl-cubed gravity is super-renormalizable.

For obtaining the classical backgrounds of this theory we should solve

$$\mathcal{G}_{\mu\nu} = 0 \quad (20)$$

which is a set of fourth order differential equations and it is not obvious how they can be integrated in generality. However, with a glimpse to $\mathcal{G}_{\mu\nu}$, one can realize two classes of solutions of (20). The first is backgrounds with $W_{\mu\nu\rho\sigma} = 0$ (conformally flat spaces) and the second is backgrounds with $F_{\mu\nu\rho\sigma} = 0$. Note also that any conformally related spaces to these two classes of solutions do also solve the (20).

3.1.1 Solutions with $W_{\rho\sigma\theta\gamma} = 0$

It is easy to see that the equation $W_{\rho\sigma\theta\gamma} = 0$ can be solved if one considers

$$R_{\rho\sigma\theta\gamma} = 2\Lambda(g_{\rho\theta}g_{\sigma\gamma} - g_{\rho\gamma}g_{\sigma\theta}).$$

As a result, geometries which are locally flat, dS or AdS are solutions of pure Weyl-cube gravity [7, 8]. Here the parameter Λ is an arbitrary constant and so, the limiting case $\Lambda = 0$ gives us the usual Schwarzschild solution.

In order to be more explicit let us consider, for simplicity, the static spherically symmetric background as

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 h(r)} + r^2 d\Omega_k^2, \quad (21)$$

where $d\Omega_k^2$ is the line element on unit 2-sphere, hyperbolic and flat spaces with $k = 1, -1, 0$ respectively. Using the above metric and doing a straightforward calculation, one can see that in order to have conformally flat space it is enough that the $f(r)$ and $h(r)$ satisfy the following equation

$$\frac{f''}{f} - 2\left(\frac{f'}{f}\right)^2 + \frac{2}{r}\frac{f'}{f} + 2\frac{f'}{f}\frac{h'}{h} - \frac{2k}{r^4h} = 0, \quad (22)$$

where the prime stands for derivative with respect to r coordinate. Then, one can integrate (22) and obtain $h(r)$ in terms of $f(r)$ as follows

$$h(r) = \frac{(4kf(r) + c)f(r)}{(r^2f'(r))^2}, \quad (23)$$

where c is a constant of integration. We see that there is degeneracy in the solution space of the theory and that there is no corresponding Birkhoff theorem in pure Weyl-cubed gravity.

As some examples, for the case where $c = k = 0$ and $f(r) = h(r) = 1$ one recovers the AdS space⁴. Moreover, setting $c = 0$ and $f(r) = (1 - \frac{M}{r})^\alpha$ then we have $h(r) = \frac{4k}{(\alpha M)^2}(1 - \frac{M}{r})^2$ which exhibits a black hole solution with horizon at $r = M$. Furthermore, setting again $c = 0$ and considering $f(r) = \frac{4}{(c_1 + c_2 r)^2}$ gives us a solution with $R = 0$ provided that $k \neq 0$. As concluding example, let us consider $f(r) = r^{2z-2}h(r)$. Then, we obtain

$$f(r) = \frac{c_1}{r^z} + \frac{c_2}{r^{2z-2}} + \frac{k}{(z-2)^2 r^2}, \quad z \neq 2, \quad (24)$$

which is asymptotically Lifschitz space only for $z = \{0, 1\}$.

At the end, it is important to note that since time dependent FRW backgrounds are conformally flat thus they are vacuum solutions of the pure Weyl-cubed gravity.

3.1.2 Solutions with $F_{\rho\sigma\theta\gamma} = 0$

In this section we want to find solutions of

$$F_{\rho\sigma\theta\gamma} = 0. \quad (25)$$

⁴Note also that, in (22) when $f(r) = \text{const}$, then $k = 0$ and so $h(r)$ is free and the solution is degenerate.

Since for any static spherically symmetric space with $F_{\rho\sigma\theta\gamma} = 0$ then we have $W_{\rho\sigma\theta\gamma} = 0$ and vice versa, we search for some metrics in which $W_{\rho\sigma\theta\gamma} \neq 0$ i.e. backgrounds with different symmetry. In particular, we examine the metric ansatz with Shrödinger or asymptotically Shrödinger symmetry. For this aim, we consider a generalized form of (9) as follows

$$ds^2 = -\frac{dt^2}{r^{2z}f(r)} + \frac{dr^2}{r^2h(r)} + 2\frac{dtdx}{r^2j(r)} + \frac{dy^2}{r^2}, \quad (26)$$

Then, the equation (25) reads as

$$\frac{j''}{j} - \left(\frac{j'}{j}\right)^2 + \frac{1}{2}\frac{j'}{j}\frac{h'}{h} = 0, \quad (27)$$

and can be solved as

$$h(r) = c\left(\frac{j(r)}{j'(r)}\right)^2. \quad (28)$$

Observe that with the solution (28) the Weyl tensor is in general nonzero⁵. Here, we have a two-fold degeneracy in the solution. Both the $f(r)$ and $j(r)$ are undetermined. For $j(r) = 1$ then (25) automatically is satisfied and by $h(r) = j(r) = 1$ we recover the Schrödinger geometry (9).

It is notable that since the pp-wave background with the line element

$$ds^2 = \frac{dr^2}{r^2h(r)} + r^2f(r)dx^2 + 2r^2j(r)dtdx + r^2dy^2, \quad (29)$$

is conformally related to Schrödinger metric, it is also the vacuum solution of pure Weyl-cubed gravity.

3.2 Einstein+Weyl-Cubed Gravity

In this section, we would like to discuss pure gravity theory in four dimensions in which includes both the Einstein term, Weyl-cubed term and also cosmological constant in the action. The equations of motion are given by

$$G_{\mu\nu} - \frac{1}{2}d(d+1)\Lambda g_{\mu\nu} + \beta\mathcal{G}_{\mu\nu} = 0. \quad (30)$$

⁵One can show that the solution (28) with $j(r) = \frac{1}{c_1 + c_2 r^{2-2z}f(r)}$ implies that $F_{\rho\sigma\theta\gamma} = W_{\rho\sigma\theta\gamma} = 0$.

Note that one may also add the curvature squared term, i.e. Gauss-Bonnet action, but in four dimensions such term is purely topological and does not contribute in the equations of motion.

In searching for classical solutions of this theory, we can classify them into two classes, those that $\mathcal{G}_{\mu\nu} = 0$ and those that $\mathcal{G}_{\mu\nu} \neq 0$. For the first class, we have classified some possible solutions of $\mathcal{G}_{\mu\nu} = 0$ in the previous sections. However, those geometries should also obey the usual Einstein equation $G_{\mu\nu} = 3\Lambda g_{\mu\nu}$. From the Einstein equation, we have $R_{\mu\nu} = -6\Lambda g_{\mu\nu}$. Therefore, backgrounds that saturate $R_{\rho\sigma\theta\gamma} = 2\Lambda(g_{\rho\theta}g_{\sigma\gamma} - g_{\rho\gamma}g_{\sigma\theta})$ are solutions of the full Einstein+Weyl-cubed gravity. As an example, by setting $\Lambda = 0$, we find the following metric

$$ds^2 = -r^{2z-2}f(r)dt^2 + \frac{dr^2}{r^4f(r)} + d\Omega_k^2, \quad (31)$$

where $f(r)$ was given in (24).

Now, we focus on the second class where $\mathcal{G}_{\mu\nu} \neq 0$. We also consider the following hyper-scaling violating static spherically symmetric metric

$$ds^2 = r^{2\alpha} \left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2d\Omega_k^2 \right), \quad (32)$$

where α exhibits the hyper-scaling violation and z is dynamical exponent. By these assumptions, however, solving the equation (30) with $\mathcal{G}_{\mu\nu} \neq 0$ is a very hard task. However, one can show that the solutions with Lifschitz asymptotic do exist only for [10]

$$\alpha = 0, -1. \quad (33)$$

Indeed, using the ansatz $f(r) = 1 - \frac{s}{r^2}$, where s is a constant, one finds the following solution(with $\mathcal{G}_{\mu\nu} \neq 0$)

$$\begin{aligned} a) \alpha = 0 \quad & s = 0, \quad k = 0, \quad \Lambda \neq 0, \quad z \neq \{0, 1, 4\} \\ & \Lambda = \frac{18 + 7z + 4z^2 - 2z^3}{9(4 - z)}, \quad \beta = -\frac{3}{2z^2(4 - 5z + z^2)}, \\ b) \alpha = -1 \quad & s \neq 0, \quad k = 0, \quad \Lambda \neq 0 \quad z = \{0, 1\} \\ & s = -\frac{(2z + 1)^2}{(z + 1)}\Lambda, \quad \beta = \frac{27}{4(z - 2)^6} \frac{\Lambda}{s^3}. \end{aligned}$$

Observe that in the case (b) we have a black hole solution with flat horizon. We can also compute the scalar invariant $W_{\rho\theta\sigma\gamma}W^{\rho\theta\sigma\gamma}$ and obtain

$$W_{\rho\theta\sigma\gamma}W^{\rho\theta\sigma\gamma} = \frac{4}{3}(z-2)^4s^2,$$

which indicates that this black hole has not curvature singularity. It is noticeable that for generating hyperscaling-violating solution in usual Einstein gravity one should consider the coupling between matter and the background metric [16] but, as we have already showed, such solution can be generated in the pure Weyl-cubed and Einstein+Weyl-cubed theories.

3.3 Cubic Gravity with Matter

In this section, we first briefly introduce the Einstein-dilaton-Maxwell theory where has been constructed in [17] and its nonlinear generalization has been studied in [18]⁶. The Lagrangian of the matter part of this theory \mathcal{L}_M in $d+1$ dimensions may be written as⁷

$$\mathcal{L}_M = -\frac{1}{2}(\partial\phi)^2 + V(\phi) + \sum_{i=1}^2 \left(-\frac{1}{4}e^{\lambda_i\phi}F_i^2 + \frac{1}{4}e^{\epsilon_i\phi}(-H_i^2)^{s_i} \right), \quad (34)$$

where $\lambda_1, \lambda_2, \epsilon_1, \epsilon_2$ are some free parameters. The field strength are defined as usual for linear gauge fields $A_{i\mu}$ as $F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}$ and for nonlinear fields $B_{i\mu}$ as $H_{i\mu\nu} = \partial_\mu B_{i\nu} - \partial_\nu B_{i\mu}$. Two constants s_i -s demonstrate the nonlinearity of the action. We will see that the presence of nonlinear field is necessary in order to solve all the equations of motion. Motivated by string theory it was considered an exponential potential for dilaton $V = V_0 e^{\gamma\phi}$ where V_0 is a free parameter [17].

Varying the action with respect to $A_{i\mu}, B_{i\mu}, \phi$ and $g_{\mu\nu}$ give us the follow-

⁶We will present the results of this section for a general $d+1$ dimensional space.

⁷Here, we write a generalized version of the action used in [17, 18]

ing equations of motion

$$\nabla_\mu (\sqrt{-g} e^{\lambda_i \phi} F_i^{\mu\nu}) = 0, \quad (35)$$

$$\nabla_\mu (s_i \sqrt{-g} e^{\epsilon_i \phi} (-H_i^2)^{s_i-1} H_i^{\mu\nu}) = 0, \quad (36)$$

$$\nabla^2 \phi + \frac{dV(\phi)}{d\phi} - \frac{1}{4} \sum_i (\lambda_i e^{\lambda_i \phi} F_i^2 - \epsilon_i e^{\epsilon_i \phi} (-H_i^2)^{s_i}) = 0, \quad (37)$$

$$\begin{aligned} R_{\mu\nu} + \beta \left(\mathcal{G}_{\mu\nu} - \frac{\mathcal{G}}{d-1} g_{\mu\nu} \right) + \\ - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{V(\phi)}{d-1} g_{\mu\nu} - \frac{1}{2} \sum_i e^{\lambda_i \phi} \left(F_{i\mu}^\rho F_{i\rho\nu} - \frac{F_i^2}{2d-2} g_{\mu\nu} \right) + \\ - \frac{1}{2} \sum_i e^{\epsilon_i \phi} \left(s_i (-H_i^2)^{s_i-1} H_{i\mu}^\rho H_{i\rho\nu} - \left(\frac{2s_i-1}{2d-2} \right) (-H_i^2)^{s_i} g_{\mu\nu} \right) = 0. \end{aligned} \quad (38)$$

where $\mathcal{G} = G^\mu{}_\mu$. We suppose the ansatz (32) with $k = 0$, and following ansatz for the scalar and gauge fields

$$\begin{aligned} ds^2 &= r^{2\alpha} \left(-r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2 \right), \\ \phi &= \phi(r), \quad F_{i\,rt} \neq 0, \quad H_{i\,rt} \neq 0. \end{aligned} \quad (39)$$

Here, we again face with a set of very difficult differential equations. Therefore, for simplicity, we search for solutions in which

$$W_{\rho\theta\sigma\gamma} = 0.$$

To proceed, first, recall that the conformally flat metrics have been given in (24). Then, since $\mathcal{G}_{\mu\nu} = 0$, we deal with the usual Einstein-dilaton-Maxwell theory. The classical solutions of this theory have been obtained in [17, 18]. Let us present the results for the more general equations (35-38).

Using the ansatz (39) and Maxwell equation one obtains

$$F_{i\,rt} = \rho_i e^{-\lambda_i \phi} r^{\alpha(3-d)+z-d} \quad (40)$$

$$H_{i\,rt} = \varrho_i e^{-\frac{\epsilon_i \phi}{2s_i-1}} r^{\frac{(2s_i-1)(z+2\alpha-1)-(d-1)(\alpha+1)}{2s_i-1}}. \quad (41)$$

Then, by focusing on $R_t^t - R_r^r$ components of equation (38) and noting that for the metric (39) we have $(R_t^t - R_r^r) \sim r^{-2\alpha} f(r)$, we are able to find the scalar field as

$$e^\phi = e^{\phi_0} r^{\sqrt{2(d-1)(\alpha+1)(\alpha+z-1)}} = e^{\phi_0} r^\Gamma. \quad (42)$$

It is clear that in order to have a well defined solution we have to consider $(\alpha + 1)(\alpha + z - 1) \geq 0$. In fact, this condition is a result of null energy condition. Indeed, one utilizes the xx components of (38) and integrate it to find $f(r)$ as follows

$$\begin{aligned}
f(r) = & - mr^{-(d-1)\alpha-z-d+1} \\
& + \frac{V_0 e^{\gamma\phi_0} r^{\gamma\Gamma+2\alpha}}{(d-1)(\alpha+1)(\gamma\Gamma+\alpha(d+1)+z+d-1)} \\
& - \sum_{i=1}^2 \frac{\rho_i^2 e^{-\lambda_i\phi_0} r^{-2\alpha(d-2)-\Gamma\lambda_i-2d+2}}{2(d-1)(\alpha+1)(\alpha(3-d)+z-d+1-\Gamma\lambda_i)} \\
& - \sum_{i=1}^2 \frac{(2\varrho_i^2)^{s_i} (2s_i-1)^2 e^{-\frac{\epsilon_i\phi_0}{2s_i-1} r^{\frac{(-2+(6-2d)s_i)\alpha-2s_i(d-1)-\Gamma\epsilon_i}{2s_i-1}}}}{4(d-1)(\alpha+1)(\alpha(4s_i-d-1)+(2s_i-1)z-d+1-\Gamma\epsilon_i)},
\end{aligned} \tag{43}$$

where m is a constant. At the next step, we focus on the equation of motion of the scalar field which can be written as

$$\begin{aligned}
\left(\frac{4\Gamma}{(d-1)(\alpha+1)} + 4\gamma \right) V_0 e^{\gamma\phi} &= \sum_{i=1}^2 e^{\lambda_i\phi} F_i^2 \left(\lambda_i - \frac{\Gamma}{(d-1)(\alpha+1)} \right) \\
&- \sum_{i=1}^2 e^{\epsilon_i\phi} (-H_i^2)^{s_i} \left(\epsilon_i - \frac{(2s_i-1)\Gamma}{(d-1)(\alpha+1)} \right),
\end{aligned} \tag{44}$$

The left and right hand sides of this equation can be equal to each other if we choose the parameters $\lambda_1, \lambda_2, \epsilon_i$ and ρ_1 as follows

$$\begin{aligned}
\lambda_1 &= -\frac{\gamma\Gamma+2(d-1)(\alpha+1)}{\Gamma}, & \lambda_2 &= \frac{\Gamma}{(d-1)(\alpha+1)} \\
\epsilon_i &= \frac{(2s_i-1)\Gamma}{(d-1)(\alpha+1)}, \\
\rho_1^2 &= 2 \frac{\Gamma+(d-1)(\alpha+1)\gamma}{\Gamma-(d-1)(\alpha+1)\lambda_1} V_0 e^{(\gamma+\lambda_1)\phi_0}.
\end{aligned} \tag{45}$$

With these relations at hand, we have several possibilities in which the (24) coincides with (39). Note that such coincidence should respect not only the remaining equations of motion but also should satisfy the null energy

condition⁸. However, for presenting a consistent solution, we first set

$$-(d-1)\alpha - z - d + 1 = 2 - 2z, \quad \gamma\Gamma + 2\alpha = -z, \quad (46)$$

which imply that

$$\alpha = \frac{z-d-1}{d-1}, \quad \Gamma = \sqrt{\frac{2d}{d-1}}(z-2). \quad (47)$$

It is clear that $(\alpha+1)(\alpha+z-1) \geq 0$ for generic value of z . We can also see that with (46) the power of r in the third line of (43) for the first linear gauge field $A_{1\mu}$ is $\gamma\Gamma + 2\alpha = -z$. Moreover, using (47) and plugging λ_2 from (45) back into the solution (43) and equating the resultant with $-z$ or $2-2z$ implies that for the second linear gauge field we should set $z=2$. Accordingly, we obtain

$$z=2, \quad \alpha=-1, \quad \Gamma=\lambda_2=0, \quad \phi=const, \quad F_{irt} \sim \frac{1}{r},$$

and the metric reduces to Minkowski metric. Therefore, let us set the free parameter $\rho_2=0$. So, we have only one linear $U(1)$ gauge field in the theory.

It is worth mentioning that if we choose $\gamma\Gamma + 2\alpha = 2-2z$ then $\rho_1=0$ so this option for the power of the r in the second line of (43) should be excluded.

In the fourth line of (43) we have two terms in which the power of r equals to $(-2 + (6-2d)s_i)\alpha - 2s_i(d-1) - \Gamma\epsilon_i$ and two free parameters ϱ_i -s. Up to now, we are free to set the powers of these terms either $-z$ or $2-2z$. But, we shall soon see that in order to satisfy the remaining equations of motion, i.e. tt and rr components of (38), one should set the power of the at least one of these terms to $-z$. Doing this for the first nonlinear gauge field $B_{1\mu}$, one obtains

$$s_1 = \frac{1}{4}, \quad (48)$$

For the second nonlinear gauge field we are yet free to set the power of r to $-z$ or $2-2z$. However, one finds that for the later case $\alpha=-1$ which is not a valid value. So the powers of r in the fourth term of (43) are also equal to $-z$ and we have $s_1=s_2=\frac{1}{4}$.

⁸In fact, the null energy condition usually rule out some other possibilities for which (24) coincides with (39).

The remaining equation of motion i.e. tt (or rr) component of (38) helps us to find a relation between the charges of nonlinear gauge fields and V_0 and ϕ_0 as follows

$$\sum_{i=1}^2 q_i^{\frac{1}{2}} e^{2\epsilon_i \phi_0} = -\frac{2^{\frac{15}{4}}}{3} V_0 e^{\gamma \phi_0} \quad (49)$$

Plugging all the results back into the (43) gives us the final form of the $f(r)$ as

$$f(r) = -mr^{2-2z}. \quad (50)$$

As we see, the contributions from the second, third and forth lines of (43) cancel each other and we have a metric with one free parameter m .

4 Conclusion

In this paper, we have studied the classical solutions of a gravitational theory where has been constructed in [5]. The action of this theory is of order three of curvature tensor. Such cubic action has been obtained by using some simple "Holographic c/a-theorem" in arbitrary dimensions. In particular, it involves only Weyl-cubed term at third order in four dimensions.

We have analyzed the solutions of the pure Weyl-cubed gravity, Einstein plus Weyl-cubed term and gravity with matter separately. In pure Weyl-cubed gravity, we classified the solutions to those that $W_{\mu\nu\lambda\theta} = 0$ and those that $W_{\mu\nu\lambda\theta} W^{\mu\nu\lambda\theta} = 0$. The former case involves the conformally flat metrics. For the later case, we obtained the Schrödinger space as a solution of pure cubic gravity in four dimensions.

After that, we have considered the full Einstein+Gauss-Bonnet+Weyl-cubed gravity. Although solving the equations of motion in this case is very hard but we found a possible solution for this theory. In particular, we supposed a background with non-zero hyper scaling violating exponent $-2\frac{\theta}{d-1}$ and dynamical exponent z and found that the solutions do exist only for $\theta = 0, 2$ in four dimensions. When $\theta = 0$, we have Lifshitz solution for any value of z except of $z = \{0, 4\}$. Moreover, when $\theta = 2$, we have black hole solution for $z = \{0, 1\}$. The above solutions exist with certain constraints on parameters of the theory.

At the end, we added the matter to the theory. The matter part includes a scalar field, linear gauge fields and nonlinear gauge fields [17, 18]. We found the conformally flat solutions of this theory i.e. $W_{\mu\nu\lambda\theta} = 0$.

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